*Journal of Global Optimization* 6: 367–381, 1995. 


# Semicoercive variational hemivariational inequalities

DANIEL GOELEVEN dgoeleven@cc.fundp.ac.be
*Chargé de Recherches au FNRS*
*Facultés Universitaires Notre Dame de la Paix*
*8, Rempart de la Vierge*
*B-5000*
*and*
*URA-CNRS 1586, Université de Limoges, France*

MICHEL THÉRA thera@cix.cict.fr
*URA-CNRS 1586*
*Université de Limoges*
*87060 Limoges Cedex, France*

**Abstract.** The aim of this paper is the mathematical study of a general class of semicoercive variational hemivariational inequalities introduced by P.D. Panagiotopoulos in order to formulate problems of mechanics involving nonconvex and nonsmooth energy function. Our approach is based on the asymptotic behavior of the functions which are involved in the variational problems.

**Keywords:** Clarke generalized derivative, variational hemivariational inequalities, recession function.

## 1. Introduction

Hemivariational inequalities have been introduced by P.D. Panagiotopoulos [15], [16] as the variational formulation of an important class of unilateral or inequality problems in Mechanics. Hemivariational inequalities then appear as a mathematical formulation of variational principles for nonconvex, nonsmooth energy problems. They are based on the concept of generalized gradient introduced by F. Clarke and on the corresponding mechanical notion of non convex superpotential [15]. It is worthwhile to notice that hemivariational and variational hemivariational inequalities have been proved very efficient to describe the behaviour of several complex structures such as multilayered plates, adhesive joints, composite structures, etc...

A complete discussion about the physical problems treated by this theory is out of scope of the present work. The interested reader is refered to [12-18] for more details concerning the applications. Our aim in this paper is only to provide necessary and sufficient conditions for the existence of solutions of a general variational hemivariational inequality.

Let $V$ be a real separable Hilbert space with topological dual $V^*$ such that

$$V \subset L^2(\Omega) \subset V^*,$$

where $\Omega$ is an open bounded and regular subset of $\mathbb{R}^n$. We assume that the injection $V \hookrightarrow L^2(\Omega)$ is dense and compact. The norms in $V$ and in $L^2(\Omega)$ are denoted by $\|\cdot\|$ and $\|\cdot\|_{L^2}$, respectively.

Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear form. We introduce the mapping $A \in L(V, V^*)$ (where $L(V, V^*)$ denotes the space of bounded linear mappings from $V$ into $V^*$) defined by

$$< Au, v >:= a(u, v), \quad \forall u, v \in V.$$

We denote by $R(A)$ the *range* of $A$, by Ker A the *kernal* of $A$, i.e., Ker A $= \{x \in V |\ A(x) = 0\}$ and by $A^*$ the corresponding *transpose* operator of $A$. It is also assumed that $a$ is semicoercive, i.e.,

$$\text{there exists} \quad c > 0 \quad \text{such that} \quad a(u, u) \ \geq \ c \cdot \|u\|^2, \quad \forall u \in \text{Ker } (A + A^*)^{\perp}.$$

identically equal to Let $f : V \to \mathbb{R}$ be a *Lipschitz* function near $x$, i.e., there exists a neighbourhood $N(x)$ of $x$ and a constant $C > 0$ depending on $x$ such that :

$$|f(u) - f(v)| \ > \ C \cdot \|u - v\|, \quad \forall u, v \in N(x).$$

Suppose $f$ is Lipschitz near $x$. Then, the *Clarke generalized directional derivative* of $f$ at $x$ in the direction $d$ is denoted by $f^o(x, d)$ and is defined by

$$f^o(x, d) := \limsup_{t \downarrow 0,\ h \to 0} \frac{f(x + h + td) - f(x + h)}{t}.$$

The *generalized gradient* of $f$ at $x$ is denoted by $\partial^C f(x)$ and is defined to be the subdifferential of the convex function $f^o(x, d)$ at 0. That is

$$\partial^C f(x) := \{w \in V^* \mid \ < w, d > \ > \ f^o(x, d), \ \forall d \in V\}.$$

Let $f$ be Lipschitz near $x$. We say that $f$ is $\partial^C$ *− regular* at $x$, if the Clarke derivative agrees with the standard directional derivative:

$$f^o(x, d) = \lim_{t \downarrow 0^+} \frac{f(x + td) - f(x)}{t}.$$

The class of $\partial^C$ *− regular* functions includes the class of convex functions and the class of maximum-type functions, i.e., functions of the type $f = \max\{\Phi_1, \cdots, \Phi_m\}$, where $\Phi_i \quad (i = 1, \cdots, m)$ are smooth functions.

Let us recall some basic well known facts about the Clarke generalized directional derivative [8]:

**Lemma 1** *(i) The function $d \to f^o(x,d)$ is finite, positively homogeneous, subadditive and then convex;*

*(ii) $\forall x \in V$, there exists a constant $C > 0$ such that :*

$$|f^o(x,d)| \;>\; C \cdot ||d||, \quad \forall d \in X;$$

*(iii) The function $d \to f^o(x,d)$ is continuous;*

*(iv) $f^o(x,-d) = (-f)^o(x,d)$;*

*(v) For each $d \in V$,*

$$f^o(x,d) = \max\{< \xi, d > \mid \xi \in \partial^C f(x)\};$$

*(vi) If $x$ is a local minimum for $f$, then*

$$0 \in \partial^C f(x);$$

*(vii) $f^0(x,d)$ is upper semicontinuous as a function of $(x,d)$.*

Throughout the paper, we suppose that the assumptions described below are satisfied :

$(\mathcal{H}_1)$ $a : V \times V \to \mathbb{R}$ is bilinear, continuous and semicoercive;

$(\mathcal{H}_2)$ $\dim \mathrm{Ker}\,(A + A^*) < +\beta$;

$(\mathcal{H}_3)$ $\Phi : V \to (-\beta, +\beta]$ is a convex lower semicontinuous functional such that $\Phi(0) = 0$;

$(\mathcal{H}_4)$ $j : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, i.e., there exists $C > 0$ such that

$$|j(x) - j(y)| \leq k|x - y|, \quad \text{for all } x, y \in \mathbb{R};$$

$(\mathcal{H}_5)$ $j(0) \in L^1(\Omega)$.

We adopt the notation " $\to$ " and " $\rightharpoonup$ " to denote the convergence with respect to the strong and the weak topology, respectively.

A variational hemivariational inequality is the problem of finding an element $u \in V$ such that

$$a(u, v-u) + \int_\Omega j^o(u, v-u)\, d\Omega \; + \Phi(v) - \Phi(u) \;\; \geq \; < f, v-u >, \;\; \forall v \in V.$$

In the rest of the paper, this problem will be denoted by $(\mathcal{P})$.

The following lemma will be useful:

**Lemma 2** *Suppose that assumptions* $(\mathcal{H}_4)$ *and* $(\mathcal{H}_5)$ *are verified. Let us denote for each* $u \in L^2(\Omega)$*:*

$$I(u) = \int_\Omega j(u)\, d\Omega.$$

*Then*

*(i)* $I$ *is defined and Lipschitz continuous on* $L^2(\Omega)$*;*

*(ii)* $I^0(u;v)$ *is weakly upper semicontinuous on* $V$ *as a function of* $(u,v)$*, i.e.,*

$$u_n \rightharpoonup u,\ v_n \rightharpoonup v \implies \limsup_{n\to+\infty} I^0(u_n;v_n) \le I^0(u;v). \tag{1}$$

*(iii)* $\int_\Omega j^0(u(\omega), v(\omega))\, d\Omega \ge I^0(u;v), \quad \forall v \in L^2(\Omega);$

*(iv)* $\int_\Omega |j^0(u(\omega), v(\omega))|\, d\Omega \le C \cdot \|v\|\ (C > 0) \quad \forall v \in V;$

*(v)* $I^0(u,\cdot)\,|_V \ge (I\,|_V)^0(u,\cdot)$.

**Proof.** (i). If $\mu$ denotes the $n$-dimensional Lebesgue's measure, then by assumption $(\mathcal{H}_4)$, we have

$$|I(u) - I(v)| \le k \cdot \sqrt{\mu}(\Omega)\|u - v\|_{L^2}, \quad \forall u, v \in L^2(\Omega).$$

By virtue of $(\mathcal{H}_5)$, then $I(0) < +\infty$ and thus $I$ is defined and Lipschitz continuous on $L^2(\Omega)$.

(ii). $I$ is locally Lipschitz and thus by Lemma 1 (6), the mapping $I^0(\cdot;\cdot)$ is upper semicontinuous on $L^2(\Omega) \times L^2(\Omega)$. Therefore, we conclude to the weak upper semicontinuity on $V \times V$ by using the compact embedding $V \hookrightarrow L^2(\Omega)$.

(iii). See [8] (Theorem 2.7.5).

(iv). We have,

$$\int_\Omega |j^0(u(\omega), v(\omega))|\, d\Omega \le \int_\Omega k \cdot |v(\omega)|\, d\Omega \le k \cdot \sqrt{\mu}(\Omega) \cdot \|v\|_{L^2}.$$

Moreover, due to the continuity of the embedding $V \hookrightarrow L^2(\Omega)$, there exists a constant $\beta > 0$ such that

$$\|v\|_{L^2} \le \beta \cdot \|v\|, \quad \forall v \in V,$$

and we get the desired result by setting $C := k\beta\sqrt{\mu}(\Omega)$.

(v). See [7]. □

Let $X$ be a real Banach space and let $G : X \to \mathbb{R} \cup +\{\infty\}$ be a functional. We call recession function of $G$ ([1]), the function

$$\begin{aligned} G_\beta(x) &:= \liminf_{\substack{t\to+\beta \\ v\to x}} \frac{G(tv)}{t} \\ &= \inf_{\substack{t_n\to\beta \\ v_n\to x}} \liminf_{n\to+\infty} \frac{G(t_n v_n)}{t_n}. \end{aligned}$$

The term "recession function" has been used previously in convex analysis [19] to denote the functional $\Psi_\infty$ associated to a convex lower semicontinuous function $\Psi : X \to \mathbb{R} \cup \{+\infty\}$ supposed to be *proper*, i.e. such that,

$$\mathrm{Dom}\,\Psi := \{x \in X \mid \Psi(x) < +\infty\}$$

is nonempty. In this case,

$$\Psi_\infty(x) := \lim_{t\to+\infty} \frac{\Psi(x_0 + tx) - \Psi(x_0)}{t}$$

where $x_0$ is taken arbitrarily such that $\Psi(x_0) < +\infty$.

In the general case, Baiocchi *et al.* [3] introduced a notion of topological recession function related to the concept of recession function associated to a nonlinear operator in the sense of Brézis and Nirenberg [4]. Recently, Attouch *et al.* [2], used another concept of recession operator in order to develop a new approach of the solvability of a generalized equation governed by a maximal monotone operator.

Finally, let us recall that the *recession cone* $K_\infty$ of a closed convex subset $K$ of $X$ is the set of those $x$ for which there exist sequences $\{t_n\}_{n\in\mathbf{N}}$ and $\{x_n\}_{n\in\mathbf{N}} \subseteq K$ such that $\lim_{n\to+\infty} t_n = +\infty$ and $x = \lim_{n\to+\infty} t_n^{-1} x_n$.

It is easy to see that $K_\beta = \mathrm{Dom}\,(\Psi_K)_\beta$, where for a convex subset $C$, $\Psi_C$ denotes the *indicator function* of $C$ and is defined by

$$\Psi_C(x) := \begin{cases} 0 & \text{if } x \in C, \\ +\infty, & \text{if } x \notin C. \end{cases}$$

These different concepts of recession funtion were widely used in [1] to prove the solvability of a general class of noncoercive, nonmonotone and nonlinear variational inequalities. In this paper, we introduce a concept of recession function associated to the Clarke generalized directional derivative of a locally Lipschitz function and we show how this notion permits to derive new results concerning the solvability of Problem $(\mathcal{P})$.

**Definition 1** *Let $j : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. We introduce and denote by $\{J^o\}_\beta$ the recession function associated to the function $G : V \to \mathbb{R}$ and* *defined by* $x \mapsto G(x) := \int_\Omega -j^o(x,-x)\,d\Omega$, *i.e.* :

$$\begin{aligned} \{J^o\}_\beta(x) &= \liminf_{\substack{t\to\beta \\ v\to x}} \int_\Omega -j^o(tv,-v)\,d\Omega \\ &= \liminf_{\substack{t\to\beta \\ v\to x}} \int_\Omega -(-j)^o(tv,v)d\Omega \end{aligned}$$

Remark that if $j$ is $\partial^c$-regular on $\mathbb{R}$, then

$$\{J^0\}_\infty(x) = \liminf_{\substack{t\to\beta \\ v\to x}} \int_\Omega j^0(tv,v)\,d\Omega.$$

## 2. Necessary conditions for the existence of solutions

The following propositions give necessary conditions for the existence of the solution.

**Proposition 1** *Assume that assumptions* $(\mathcal{H}_1)$ *through* $(\mathcal{H}_5)$ *are satisfied. Let* $u$ *be a solution of Problem* $(\mathcal{P})$.

*(i) Then*

$$< f,q >> \int_\Omega j^o(u,q)\,d\Omega + \Phi_\infty(q), \quad \forall q \in \text{Ker } A.$$

*Moreover, there exists a constant* $c > 0$ *such that :*

$$< f,q >> |\Phi_\infty(q)| + c \cdot ||q||_{L^1}, \quad \forall q \in \text{Ker } A,$$

*where* $||\cdot||_{L^1}$ *denotes the norm in* $L^1(\Omega)$,

*(ii) If*

*(A)* $\Phi_\infty$ *is odd*

*(B)* $j$ *is* $\overline{\partial}$*-regular*

*then*

$$< f,q >= \int_\Omega j^o(u,q)\,d\Omega + \Phi_\infty(q), \quad \forall q \in \text{Ker } A.$$

*(iii) Let* $K$ *be a nonempty closed convex set. If* $\Phi$ *is the indicator function* $\Psi_K$ *of a closed convex set* $K$, *then*

$$\int_\Omega j^o(u,q)\,d\Omega \geq < f,q >, \quad \forall q \in K_\beta \cap \text{Ker } A.$$

*Moreover, there exists a constant $c > 0$ such that :*

$$< f, q > > \ c \cdot ||q||_{L^1}, \quad q \in K_\beta \cap \text{Ker } A.$$

**Proof.** (i). Let $u$ be a solution of $(\mathcal{P})$. Then

$$a(u, v-u) + \int_\Omega j^o(u, v-u)\, d\Omega + \Phi(v) - \Phi(u) \ \geq < f, v-u >, \quad \forall v \in V.$$

Let $q \in \text{Ker } A$ and set $v := u + q$. We have

$$\int_\Omega j^o(u, q)\, d\Omega + \Phi(u+q) - \Phi(u) \ \geq < f, q >, \forall q \in \text{Ker } A.$$

Thus, [5], [Proposition 1, ii]

$$\int_\Omega j^o(u, q) d\Omega + \Phi_\infty(q) \ \geq < f, q >, \quad \forall q \in \text{Ker } A.$$

Moreover

$$< f, q > > \left| \int_\Omega j^o(u, q) d\Omega \right| + |\Phi_\infty(q)|.$$

By assumption $(\mathcal{H}_4)$

$$|j^o(u, q)| \ > \ k|q|,$$

and thus

$$< f, q > > \ k \cdot ||q||_{L^1} + |\Phi_\infty(q)|.$$

(ii) By (i) we have also

$$\int_\Omega j^o(u, -q)\, d\Omega + \Phi_\infty(-q) \ \geq \ - < f, q >, \quad \forall q \in \text{Ker } A.$$

Since $\Phi_\infty$ is odd, then $\Phi_\infty(-q) = -\Phi_\infty(q)$. Combining the facts that by Lemma 1 (iv), $j^o(u, -q) = (-j)^o(u, q)$ and that if $j$ is $\overline{\partial}$-regular then $(-j)^o(u, q) = -j^o(u, q)$,we obtain:

$$\int_\Omega j^o(u, q)\, d\Omega + \Phi_\infty(q) \ > < f, q >, \quad \forall q \in \text{Ker } A. \tag{2}$$

Thus, (2) combine to (i), yields (ii).

(iii). If $\Phi$ is the indicator function of a nonempty closed convex subset $K$, then $u \in K$ and

$$a(u, v-u) + \int_\Omega j^o(u, v-u)\, d\Omega \ \geq < f, v-u >, \quad \forall v \in K.$$

If $q \in K_\infty$ then $v := u + q \in K$, so that

$$\int_\Omega j^o(u,q)\ d\Omega \ \geq < f, q >, \quad \forall q \in K_\infty \cap \operatorname{Ker} A. \qquad \square$$

Let $\beta \in L^\infty_{loc}(\mathbb{R})$ and suppose that $j$ is obtained from $\beta$ by a simple integration, i.e.,

$$j(\xi) := \int_0^\xi \beta(t)dt. \tag{3}$$

We assume that there exists two constants $\beta_- \in \mathbb{R} \cup \{-\infty\}$ and $\beta_+ \in \mathbb{R} \cup \{+\infty\}$ such that

$$\beta_- \ > \ \beta(\xi) \ > \ \beta_+, \quad \forall \xi \in \mathbb{R}.$$

We have the following result:

**Proposition 2** *Let $j$ be defined by (3). Let $u, q \in V$, we have :*

$$\int_\Omega \beta_- \cdot q^+ - \beta_+ \cdot q^- \ d\Omega \ > \ \int_\Omega j^o(u,q)\, d\Omega \ > \ \int_\Omega \beta_+ \cdot q^+ - \beta_- \cdot q^-\, d\Omega,$$

*where $q^+ := \sup\{0, q\}$ and $q := \sup\{0, -q\}$.*

**Proof.**

$$\begin{aligned}
\int_\Omega j^o(u,q)\ d\Omega \ &= \ \int_\Omega \left( \limsup_{\lambda \to 0+,\, h \to 0} 1/\lambda \int_{u+h}^{u+h+\lambda q} \beta(t)dt \right) d\Omega \\
&= \int_{\Omega+} \left( \limsup_{\lambda \to 0+, h \to 0} \ 1/\lambda \int_{u+h}^{u+h+\lambda q} \beta(t)dt \right) d\Omega \\
&+ \int_{\Omega-} \left( \limsup_{\lambda \to 0+, h \to 0} \ 1/\lambda \int_{u+h+\lambda q}^{u+h} -\beta(t)dt \right) d\Omega
\end{aligned}$$

where

$$\Omega^+ := \{x \in \Omega |\, q(x) \ > \ 0\} \text{ and } \Omega^- := \{x \in \Omega |\, q(x) < 0\}.$$

It is then easy to see that

$$\begin{aligned}
\int_\Omega j^o(u,q)\ d\Omega \ &> \ \int_{\Omega^+} \beta_+ \cdot q\, d\Omega + \int_{\Omega^-} \beta_- \cdot q\, d\Omega \\
&= \ \int_\Omega \beta_+ \cdot q^+ - \beta_- \cdot q^-\ d\Omega.
\end{aligned}$$

Similarly, we prove that

$$\int_\Omega \beta_- \cdot q^+ - \beta_+ \cdot q^-\ d\Omega \ > \ \int_\Omega j^o(u,q)\, d\Omega. \qquad \square$$

Combining Proposition 1 and Proposition 2, we derive the following result:

**Proposition 3** *Let $j$ be defined by (3). If $u$ is a solution of Problem $(\mathcal{P})$ then the following relations hold true.*

*(i)*

$$\int_\Omega \beta_+ \cdot q^+ - \beta_- \cdot q^- \, d\Omega + \Phi_\infty(q) \ \geq < f, q >, \ \forall q \in \text{Ker } A;$$

*(ii) If $\Phi_\infty$ is odd, then*

$$\int_\Omega \beta_- \cdot q^+ - \beta_+ \cdot q^- \, d\Omega + \Phi_\infty(q) \ > < f, q > > \ \Phi_\infty(q) + \int_\Omega \beta_+ \cdot q^+ - \beta_- \cdot q^- \, d\Omega, \quad \forall q \in \text{Ker}$$

*(iii) Let $K$ be a nonempty closed convex set. If $\Phi := \Psi_K$ then*

$$\int_\Omega \beta_+ \cdot q^+ - \beta_- \cdot q^- \, d\Omega \ \geq < f, q >, \ , \forall q \in K_\infty \cap \text{Ker } A.$$

**Remark 1** Let us define

$$\beta(-\infty) := \limsup_{\xi \to -\infty} \beta(\xi),$$

and

$$\beta(+\infty) := \liminf_{\xi \to +\infty} \beta(\xi).$$

Suppose that

$$\beta(-\infty) \ > \ \beta(\xi) \ > \ \beta(+\infty), \quad \forall \xi \in \mathbb{R}.$$

Then a necessary condition for the existence of a solution $u \in V$ for Problem $(\mathcal{P})$ when $\Phi \equiv 0$ is the following one which had already been obtained by P.D. Panagiotopoulos [13]:

$$\int_\Omega \beta(-\infty) \cdot q^+ - \beta(+\infty) \cdot q^- \, d\Omega \ > < f, q > > \ \int_\Omega \beta(+\infty) \cdot q^+ - \beta(-\infty) \cdot q^- \, d\Omega, \quad \forall q \in \text{Ker } A.$$

## 3. Sufficient conditions for the existence of solutions

**Theorem 1** *Assume that assumption $(\mathcal{H}_1)$ through $(\mathcal{H}_5)$ are satisfied. If*

$$\{J^o\}_\infty(w) + \Phi_\infty(w) \ > < f, w >, \quad \forall w \in \text{Ker } (A + A^*) \backslash \{0\},$$

*then Problem $(\mathcal{P})$ has at least one solution.*

**Proof.** Part 1. Let $C$ be a nonempty weakly compact subset of $V$. Suppose that $0 \in C$ and define

$$g(u, v) := a(u, u - v) - I\,|^{o}_{V}(u, v - u) + \Phi(u) - \Phi(v) - < f, u - v > .$$

We remark that :

(a) $\Phi(0) = 0 < +\infty$

(b) $g(u,u) > 0$, for all $u \in C$,

(c) $g(u,\cdot)$ is concave for all $u \in C$.

Using assumptions $(\mathcal{H}_1)$ through $(\mathcal{H}_5)$ and Lemma 2 ii), then $g(\cdot, v)$ is weakly lower semicontinuous on $V$ for all $v \in C$. Thus, by the Ky-Fan inequality [11], there exists $u^* \in C$ such that

$$g(u^*, v) > 0, \quad \forall v \in C, \ i.e.,$$

$$a(u^*, v - u^*) + I\,|_V^0(u^*, v - u^*) + \Phi(v) - \Phi(u^*) \geq \langle f, v - u^* \rangle, \quad \forall v \in C.$$

Thus, by Lemma 2 (iii) and (v), the following relation is satisfied for all $v \in C$:

$$a(u^*, v - u^*) + \int_\Omega j^o(u^*, v - u^*) d\Omega + \Phi(v) - \Phi(u^*) \geq < f, v - u^* > . \qquad (4)$$

Part 2. Let define

$$R: = \{w \in X |\, \exists u_n \in V, \ t_n := ||u_n|| \to +\infty, \ w_n := u_n/||u_n|| \rightharpoonup w \text{ and}$$

$$a(u_n, u_n) - \int_\Omega j^o(u_n, -u_n)\, d\Omega + \Phi(u_n) > < f, u_n >\}.$$

We will prove that if $R$ is empty then Problem $(\mathcal{P})$ has at least one solution.

If $B_n := \{v \in V |\, ||v|| > n\}$, by applying Part 1, there exists $u_n \in B_n$ such that

$$a(u_n, v - u_n) + \int_\Omega j^o(u_n, v - u_n)\, d\Omega + \Phi(v) - \Phi(u_n) \geq < f, v - u_n >, \quad \forall v \in B_n.$$

We prove that $||u_k|| < k$ for some integer $k$. Indeed suppose on the contrary that $||u_n|| = n$, for each $n \in \mathbb{N} \setminus \{0\}$. On relabeling if necessary, the sequence defined by $w_n := n^{-1} u_n \rightharpoonup w$ and satisfies

$$a(u_n, u_n) - \int_\Omega j^o(u_n, -u_n)\, d\Omega + \Phi(u_n) > < f, u_n > .$$

This implies $w \in R$ and a contradiction.

We prove now that $u_k$ solves $(\mathcal{P})$. Indeed, for each $y \in V$, there exists $\varepsilon > 0$ such that $u_k + \varepsilon(y - u_k) \in B_k$. Take $\varepsilon < [k - ||u_k||]/[||y - u_k||]$ if $y \alpha u_k$ and $\varepsilon = 1$ if $y = u_k$ and put $v = u_k + \varepsilon(y - u_k)$ in (4). Then we have

$$\varepsilon \cdot a(u_k, y - u_k) + \varepsilon \cdot \int_\Omega j^o(u_k, y - u_k)\, d\Omega + \Phi(u_k + \varepsilon(y - u_k)) - \Phi(u_k) \geq \varepsilon \cdot < f, y - u_k > .$$

Hence, by using the convexity of $\Phi$, we derive

$$a(u_k, y - u_k) + \int_\Omega j^o(u_k, y - u_k)\, d\Omega + \Phi(y) - \Phi(u_k) \geq < f, y - u_k >, \ \forall y \in V.$$

Part 3. It remains to prove that $R$ is empty. Suppose on the contrary, there exists a sequence $\{u_n\}_{n\in\mathbf{N}}$ such that $u_n \in V$, $t_n := \|u_n\| \to \infty$, $w_n := \|u_n\|^{-1}u_n \rightharpoonup w$, and

$$a(u_n, u_n) - \int_\Omega j^o(u_n, -u_n)\, d\Omega + \Phi(u_n) \; > \; < f, u_n > . \tag{5}$$

Since $\Phi$ is a proper lower semicontinuous function, there exists $\alpha_1 \geq 0$ and $\alpha_2 \in \mathbb{R}$ such that

$$\Phi(x) \geq -\alpha_1\|x\| - \alpha_2, \qquad \forall x \in V.$$

We have

$$\liminf_{n\to+\infty} a(w_n, w_n) + \liminf_{n\to+\infty} - \int_\Omega j^o(u_n, -u_n)/t_n^2\, d\Omega + \liminf_{n\to+\infty}(-\alpha_1\|u_n\| - \alpha_2)/t_n^2 \; > \; 0.$$

By Lemma 2 (iv), we have

$$\left|\int_\Omega j^o(u_n, -u_n)/t_n^2\, d\Omega\right| \; > \; c\cdot\|u_n\|/t_n^2 \; > \; c\cdot\|w_n\|/t_n \; = \; c/t_n.$$

Thus

$$-c/t_n \; > \; -\int_\Omega j^o(u_n, -u_n)/t_n^2\, d\Omega \; > \; c/t_n,$$

so that

$$\liminf_{n\to+\infty} - \int_\Omega j^o(u_n, -u_n)/t_n^2\, d\Omega = 0.$$

Thus $\liminf\limits_{n\to+\infty} a(w_n, w_n) = 0$, and on relabeling if necessary, we can suppose that $w_n \to w \in \text{Ker } (A + A^*)$, $\|w\| = 1$ (see [10] for more details). Consider (5) for this subsequence, and divide by $t_n$. We obtain

$$\liminf_{n\to+\infty} a(t_n w_n, w_n) + \liminf_{n\to+\infty} - \int_\Omega j^o(t_n w_n, -w_n)\, d\Omega + \liminf_{n\to+\infty} \Phi(t_n w_n)/t_n \; > \; < f, w_n >,$$

and thus

$$\{J^o\}_\infty(w) + \Phi_\infty(w) \; > \; < f, w >,$$

which is a contradiction. $\square$

**Proposition 4** *Let $j$ be defined by (3). We assume that*

*(a)* $-\infty < \beta(-\infty) \; > \; \beta(\xi) \; > \; \beta(+\infty) < +\infty, \quad \forall \xi \in \mathbb{R}$,

*(b)* $\beta \in C^o(\mathbb{R}\backslash B(0, R))$, *for some* $R > 0$, *with* $B(0, R) := \{x \in \mathbb{R} | \; |x| \; > \; R\}$.

*Then*

*(i) $j$ is Lipschitz continuous;*

*(ii)* $j(0) \in L^1(\Omega)$;

*(iii)* $\{J^o\}_\infty(w) \ \geq \ \int_\Omega \beta(+\infty)\cdot q^+ - \beta(-\infty)\cdot q^- \, d\Omega.$

**Proof.**

Part (i). Let $x, y \in \mathbb{R}$. Clearly,

$$|j(x) - j(y)| \ > \ \max\{|\beta(-\infty)|, |\beta(+\infty)|\}\cdot |x-y|.$$

Part (ii). By (3), $j(\xi) = \int_0^\xi \beta(t)dt$. Thus, $\int_\Omega j(0)\, d\Omega = 0$. As a result, $j(0) \in L^1(\Omega)$.

Part (iii).

$$\begin{aligned}
\{J^0\}_\infty(w) &= \inf_{\substack{q_n\to q\\ t_n\to+\infty}} \liminf_{n\to+\infty} \int_\Omega -j^0(t_nq_n; -q_n) \\
&= \inf_{\substack{q_n\to q\\ t_n\to+\infty}} \liminf_{n\to+\infty} \int_\Omega - \limsup_{\substack{\lambda\to0+\\ h\to0}} -1/\lambda \left( \int_{t_nq_n+h}^{t_nq_n+h+\lambda q_n} \beta(r)dr \right) d\Omega \\
&= \inf_{\substack{q_n\to q\\ t_n\to+\infty}} \liminf_{n\to+\infty} \int_\Omega \liminf_{\substack{\lambda\to0+\\ h\to0}} \left( 1/\lambda \int_{t_nq_n+h}^{t_nq_n+h+\lambda q_n} \beta(r)dr \right) d\Omega.
\end{aligned}$$

Let $q_n \in V$, $t_n \to \infty$ such that $q_n \to q$ in $V$, $\int_\Omega -\{-j\}^o(t_n\cdot q_n, q_n)\ d\Omega \to \{J^o\}_\infty(w)$. Since $V$ is compactly imbedded in $L^2(\Omega)$, $q_n \to q$ in $L^2(\Omega)$. Extracting a subsequence, we can always assume that $q_n(x) \to q(x)$ a.e. on $\Omega$ and that there exists some fixed function $h \in L^2(\Omega)$ such that $|q_n| \ > \ h$, for each $n$.

Let define

$$f_n := \liminf_{\lambda\to0^+, h\to0} 1/\lambda \int_{t_nq_n+h}^{t_nq_n+h+\lambda q_n} \beta(\tau)d\tau.$$

It suffices to observe that

$$f_n \leq \max\{|\beta(-\infty)|, |\beta(+\infty)|\}\cdot|h|.$$

Indeed, if $q_n \ \geq \ 0$, then $|f_n| \ > \ |\beta(+\infty)|\cdot|h|$, and if $q_n < 0$, then $|f_n| \ > \ |\beta(-\infty)|\cdot|h|$. A simple application of the Lebesgue dominated convergence Theorem then yields:

$$\begin{aligned}
&\liminf_{n\to\infty} \int_\Omega \liminf_{\lambda\to0^+,h\to0} 1/\lambda \int_{t_nq_n+h}^{t_nq_n+h+\lambda q_n} \beta(\tau)\, d\tau d\Omega \\
&> \int_\Omega \liminf_{n\to\infty,\lambda\to0^+,h\to0} 1/\lambda \int_{t_nq_n+h}^{t_nq_n+h+\lambda q_n} \beta(\tau)d\tau d\Omega.
\end{aligned}$$

Set

$$\Omega^+ := \{x \in \Omega |\, q(x) > 0\},$$
$$\Omega^- := \{x \in \Omega |\, q(x) < 0\},$$

and

$$\Omega^0 := \{x \in \Omega |\, q(x) = 0\}.$$

- On $\Omega^+$ and for $n$ large enough, by the Mean Value Theorem we have,

$$\int_{t_n q_n + h}^{t_n q_n + h + \lambda q_n} \beta(\tau) d\tau = \beta(\zeta_n) \lambda q_n,$$

for some $\zeta_n \in [t_n q_n + h, t_n q_n + h + \lambda q_n]$. Since $\zeta_n \to \infty$,we derive

$$\int_{\Omega^+} \liminf_{\substack{n\to\infty \\ \lambda\to 0^+,\, h\to 0}} 1/\lambda \int_{t_n q_n + h}^{t_n q_n + \lambda q_n} \beta(\tau) d\tau\, d\Omega \;\geq\; \int_{\Omega^+} \beta(+\infty)\cdot q d\Omega = \int_\Omega \beta(+\infty)\cdot q^+ d\Omega.$$

- On $\Omega^-$ and for $n$ large enough :

$$\int_{t_n q_n + h + \lambda q_n}^{t_n q_n + h} -\beta(\tau) d\tau = \beta(\xi_n) \lambda q_n,$$

for some $\xi_n \in [t_n q_n + h + \lambda q_n, t_n q_n + h]$. Since $\xi_n \to -\infty$, we derive

$$\int_{\Omega^-} \liminf_{\substack{n\to\infty \\ \lambda\to 0^+,\, h\to 0}} 1/\lambda \int_{t_n q_n + h}^{t_n q_n + h + \lambda q_n} \beta(\tau) d\tau\, d\Omega \;\geq\; \int_{\Omega^-} \beta(-\infty)\cdot q \, d\Omega = -\int_\Omega \beta(-\infty)\cdot q^- \, d\Omega.$$

- On $\Omega^0$:

$$\left| \int_{\Omega^0} \liminf_{\substack{n\to\infty \\ \lambda\to 0^+,\, h\to 0}} 1/\lambda \int_{t_n q_n + h}^{t_n q_n + h + \lambda q_n} \beta(\tau) d\tau\, d\Omega \right|$$

$$> \max\{|\beta(-\infty)|, |\beta(+\infty)|\} \cdot \int_{\Omega^O} \liminf_{n\to\infty} |q_n| \, d\Omega = 0.$$

Thus

$$\{J^0\}_\infty(w) \;\geq\; \int_\Omega \beta(+\infty)\cdot q^+ - \beta(-\infty)\cdot q^- d\Omega. \qquad \square$$

**Corollary 1** *Let $j$ be defined by (3). We assume that*

*(i)* $-\infty < \beta(-\infty) \;>\; \beta(\xi) \;>\; \beta(+\infty) < +\infty, \forall \xi \in \mathbb{R}$,

*(ii)* $\beta \in C^o(\mathbb{R} \backslash B(0,R))$, *for some* $R > 0$.

*Assume that assumptions $(\mathcal{H}_1)$ through $(\mathcal{H}_3)$ are satisfied*

*If*

$$(\mathcal{A}) \quad \int_{\Omega} \beta(+\infty)\cdot q^{+} - \beta(-\infty)\cdot q^{-} \, d\Omega + \Phi_{\infty}(w) \; > \; < f, w >, \forall w \in \text{Ker } (A+A^{*})\backslash\{0\},$$

*then Problem* $(\mathcal{P})$ *has at least one solution.*

**Remark 2** (i) If $\Phi$ is positive and positively homogeneous of order 1 then $\Phi_{\infty}(w) = \Phi(w)$.

(ii) If $\Phi$ is positive and positively homegeneous of order $\alpha > 1$ then $\Phi_{\infty}(w) = \Psi_{\text{Ker } \Phi}(w)$ and condition (A) reads as follows :

$$(A') \quad \int_{\Omega} \beta(+\infty)\cdot q^{+} - \beta(-\infty)\cdot q^{-} \, d\Omega \; > \; < f, w >, \quad \forall w \in \text{Ker } (A+A^{*}) \cap \text{Ker } \Phi\backslash\{0\}.$$

(iii) If $\Phi$ is positively homegeneous of order $1 > \alpha > 0$ then $\Phi_{\infty}(w) = \Psi_{\text{Dom } \Phi}(w)$ and condition (A) reduces to the following form :

$$(A'') \quad \int_{\Omega} \beta(+\infty)\cdot q^{+} - \beta(-\infty)\cdot q^{-} \, d\Omega \; > \; < f, w >, \quad \forall w \in \text{Ker } (A+A^{*}) \cap \text{Dom } \Phi\backslash\{0\}.$$

## References


1. S. Adly, D. Goeleven and M. Théra, *Recession Mappings and Noncoercive Variational Inequalities*, to appear in Nonlinear Analysis, Theory, Methods and Applications.
2. H. Attouch, Z. Chbani and A. Moudafi, *Recession Operators and Solvability of Variational Problems*, Preprint, Laboratoire d'Analyse Convexe, Université de Montpellier, 1993.
3. C. Baiocchi, G. Buttazzo, F. Gastaldi and F. Tomarelli, *General Existence Theorems for Unilateral Problems in Continum Mechanics*, Arch. Rat. Mech. Anal. **100**, 2, pp. 149-180, 1988.
4. H. Brézis and L. Nirenberg, *Characterizations of the Ranges of Some Nonlinear Operators and Applications to Boundary Value Problems*, Ann. Scuola Normale Superiore Pisa, Classe di Scienze Serie IV, vol. **V**, N°**2**, pp. 225-235, 1978.
5. G. Bouchitte & P. Suquet, *Equicoercivity of variational problems: the role of recession functions*, preprint.
6. J.C. Burbill, *The Lebesgue Integral*, Cambridge University Press 1963.
7. K.-C. Chang, *Variational Methods for Nondifferentiable Functionals and their Applications to Partial Differential Equations*, J. Math. Anal. and Appl., **80**, 102-129, 1981.
8. F. H. Clarke, *Nonsmooth Analysis and Optimization*, Wiley, New York 1984.
9. J. Dieudonné, *Eléments d'Analyse*, Gauthier-Villars, 1968.
10. D. Goeleven, *On the solvability of noncoercive linear variational inequalities in separable Hilbert Spaces*, Journal of Optimization Theory and Applications, Vol. **79**, N°3, pp. 493-511, 1993.
11. U. Mosco, *Implicit Variational Problems and Quasi-Variational Inequalities*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, Germany, Vol. **543**, pp. 83-156, 1976.
12. P.D. Panagiotopoulos, *Coercive and Semicoercive Hemivariational Inequalities*, Nonlinear Analysis, Theory, Methods and Applications, Vol. **16**, N°3, pp. 209-231, 1991.
13. P.D. Panagiotopoulos, *Inequality Problems in Mechanics and Applications*, Birkhduser 1985.

14. P.D. Panagiotopoulos, *Hemivariational Inequalities and their Applications*, In : J.J. Moreau, P.D. Panagiotopoulos, G. Strang (eds), *Topics in Nonsmooth Mechanics*, Birkha—ser Verlag 1988.
15. P.D. Panagiotopoulos, *Nonconvex Superpotentials in the Sense of F.H. Clarke and Applications*, Mech. Res. Comm. **8**, pp. 335-340, 1981.
16. P.D. Panagiotopoulos, *Nonconvex Energy Function, Hemivariational Inequalities and Substationarity Principles*, Acta Mech. **48**, pp. 160-183, 1983.
17. P.D. Panagiotopoulos, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer Verlag Berlin Heidelberg 1993.
18. P.D. Panagiotopoulos and G.E. Stavroulakis, *A Variational-Hemivariational Inequality Approach to the Laminated Plate Theory Under Subdifferential Boundary Conditions*, Quaterly of Applied Mathematics, Vol. **XLIV**, N°3, pp. 409-430, 1988.
19. R.T. Rockafellar, *Convex Analysis, Princeton*, University Press, Princeton N.J., 1970.